Error estimates of the Non-Intrusive Reduced Basis 2-grid method with parabolic equations

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Reduce the computational costs of parameter-dependent problems with Non-Intrusive Reduced Basis methods
Figure: Solution manifold

\[ \mathcal{M} = \{ u(\mu) \in V \mid \mu \in G \} \subset V. \]

- Parameter: \( \mu \in G \),
- Solution: \( u(\mu) \in V \).
Introduction to the NIRB methods

Reduced basis methods

\[ \mathcal{M} = \{ u(\mu) \in V \mid \mu \in \mathcal{G} \} \subset V. \]

- \( X^N \) Reduced basis space,
- Parameters \( \mu_1, \ldots, \mu_N \in \mathcal{G} \),
- Snapshots \( u(\mu_1), \ldots, u(\mu_N) \in V_h \),
- Projected snapshots onto \( X^N \).
- Projected new solution onto \( X^N \).

Figure: Solution manifold
Introduction to the NIRB methods

Reduced basis methods

\[ \mathcal{M} = \{ u(\mu) \in V \mid \mu \in \mathcal{G} \} \subset V. \]

- \( X^N \) Reduced basis space,
- Parameters \( \mu_1, \ldots, \mu_N \in \mathcal{G}, \)
- Snapshots \( u(\mu_1), \ldots, u(\mu_N) \in V, \)
- Projected snapshots onto \( X^N. \)

For the solution manifold \( \mathcal{M} \), the infimum of the distance \( \inf_{\dim(X^N)=N} \text{dist}(\mathcal{M}, X^N) \) must be small.\(^1\)\(^2\)

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Introduction to the NIRB methods

**Reduced basis methods**

\( \mathcal{M}_h = \{ u_h(\mu) \in V_h \mid \mu \in \mathcal{G} \} \subset V_h \).

- \( X_h^N \) Reduced basis space,
- Parameters \( \mu_1, \ldots, \mu_N \in \mathcal{G} \),
- Snapshots \( u_h(\mu_1), \ldots, u_h(\mu_N) \in V_h \),
- Projected snapshots onto \( X_h^N \).

Figur: Solution manifold

Kolmogorov n-width must be small \(^1\) \(^2\)

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Introduction to the NIRB methods

Reduced basis methods

- Optimization over parameter space
- High Fidelity (HF) real-time simulations

Non-Intrusive Reduced basis methods (NIRB)

Industrial context → black box solver
Introduction to the two-grid method within the parabolic context
A model problem

\[
\begin{align*}
\begin{cases}
  u_t - \mu \Delta u &= f, \quad \text{in } \Omega \times ]0, T], \\
  u(x, 0) &= u_0(x), \quad \text{in } \Omega, \\
  u &= 0, \quad \text{on } \partial \Omega, \quad \forall t \in [0, T],
\end{cases}
\end{align*}
\]

- \( \mu \in \mathbb{R} \): Variable parameter
- \( u(x, t; \mu) \): Unknowns
  - \( u^n_h \in V_h \) on the fine mesh \( T_h \) and fine time grid \( F_n \) (HF),
  - \( u^m_H \in V_H \) on the coarse mesh \( T_H \) and coarse time grid \( G_m \).

1. Offline stage: \( u_h((\mu, t^n)_i) \): Snapshots on \( T_h \)
2. Online stage: \( u_H(\mu, \tilde{t}^m) \): Solution on \( T_H \) \( (H^2 \sim h) \)

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5. E. Grosjean, Y. Maday, A doubly reduced approximation for the solution to PDE’s based on a domain truncation and a reduced basis method: Application to Navier-Stokes equations. 2022.
The NIRB two-grid method is applied with two different time schemes.
Decomposition

Separation of variables

\[ u_h(x, t; \mu) = \sum_{j=1}^{N} a_j^h(\mu, t^n) \Phi_j^h(x), \]

\((\Phi_j^h)_{j=1, \ldots, N} \in X_h^N : L^2\)-orthonormalized basis functions (modes)

**Coefficients** \(a_j^h(\mu, t^n)\)

- Optimal coefficients: \((u_h(\mu, t^n), \Phi_j^h(x))\),
- Our choice: \((u_H(\mu, \tilde{t}^m), \Phi_j^h(x))\), with \((\Phi_j^h)_{j=1, \ldots, N} L^2 \& H^1\)-orthogonalized
NIRB – OFFLINE/ONLINE

**OFFLINE**
- **FINE SNAPSHOTS** \( \{u_h(\mu_1, t^1), \ldots, u_h(\mu_1, t^n)\} \)
- **COARSE SNAPSHOTS** \( \{u_H(\mu_1, \tilde{t}^1), \ldots, u_H(\mu_{N_{\text{train}}}, \tilde{t}^m)\} \)
- **REDUCED BASIS** \( (\Phi_i^h)_{i=1, \ldots, N} \)
- **RECTIFICATION** \( R^n \)

**ONLINE**
- **COARSE SOLUTION** \( u_H(\mu, \tilde{t}^m) \)

Projection:
\[
\sum_{i=1}^{N} (I_h(u_H(\mu, \tilde{t}^m)), \Phi_i^h) \Phi_i^h
\]

Projection with rectification:
\[
\sum_{i,j=1}^{N} R^n_{i,j} (I_h(u_H(\mu, \tilde{t}^m)), \Phi_j^h) \Phi_i^h
\]

NIRB approximation:
\[
u_{Hh}^N(u, \mu)
\]
Greedy algorithm

→ $L^2$ orthonormalization.
+ Eigenvalue problem: $\forall \nu \in X^N_h, \int_\Omega \nabla \Phi_h \cdot \nabla \nu = \lambda \int_\Omega \Phi_h \cdot \nu$
→ $L^2(\Omega)$ and $H^1(\Omega)$ orthogonalization.

$X^N_h = \text{Span}\{\phi^h_1, \ldots, \phi^h_N\}$
Greedy algorithm

for \( k = 1, \ldots, N \):

\[
\tilde{\mu}_k = \arg \max_{\mu \in \mathcal{G}, n=0,\ldots,\frac{T}{\Delta t_F}} \frac{\| u_h(\mu, t^n) - P^k-1(u_h(\mu, t^n)) \|}{\| u_h(\mu, t^n) \|}
\]

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Energy error estimate with $P_1$ FE (parabolic equations)

\[ \forall n, \left\| u(t^n)(\mu) - u_{Hh}^{n, n}(\mu) \right\|_{H^1(\Omega)} \leq \varepsilon(N) + C_1 h + C_2 \Delta t_F + C_3 (N) H^2 + C_4 (N) \Delta t_G^2, \]

\[ \sim O(h) + O(\Delta t_F) \text{ if } H^2 \sim h \text{ and } \Delta t_G^2 \sim \Delta t_F, \]

where $C_1, C_2$ are constants independent of $h$ and $H$. \(^7\)

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\(^7\) V. Thomee. *Galerkin finite element methods for parabolic problems*. 2007
FEM Error estimates

Energy error estimate with $P_1$ FE (parabolic equations)

$$\forall n, \left\| u(t^n)(\mu) - u_{Hh}^{N,n}(\mu) \right\|_{H^1(\Omega)} \leq \underbrace{\varepsilon(N)}_{T_1} + \underbrace{C_1 h + C_2 \Delta t_F}_{T_2} + \underbrace{C_3(N) H^2 + C_4(N) \Delta t_G^2}_{T_3},$$

$$\sim O(h) + O(\Delta t_F) \text{ if } H^2 \sim h \text{ and } \Delta t_G^2 \sim \Delta t_F,$$

where $C_1, C_2$ are constants independent of $h$ and $H$. \(^7\)

Crank-Nicholson $L^2$ estimate ($P_1$ FE).

$$\forall m \geq 0,$$

$$\left\| u(t^m) - u_H^m \right\|_{L^2(\Omega)} \leq C H^2 \left[ \left\| u_0 \right\|_{H^2(\Omega)} + \int_0^{t^m} \left\| u_t \right\|_{H^2(\Omega)} \ ds \right] + C \Delta t_G^2 \int_0^{t^m} \left( \left\| u_{ttt} \right\|_{L^2(\Omega)} + \left\| \Delta u_{tt} \right\|_{L^2(\Omega)} \right) \ ds.$$

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\(^7\) V. Thomee. *Galerkin finite element methods for parabolic problems*. 2007
NIRB
Elise Grosjean

Introduction
A model problem
Error estimates
Numerical results

Numerical results with FEM

\[ f(t, x) = 10[x^2(x - 1)^2y^2(y - 1)^2 - 2((6x^2 - 6x + 1)(y^2(y - 1)^2) + (6y^2 - 6y + 1)(x^2(x - 1)^2))] \]
\[ \mu \in (0, 10] \]

Relative errors with NIRB algorithm

Figure: Test with \( N_{\text{train}} = 10, \mu = 1, h \simeq \Delta t_F \)
Numerical results with FEM

\[ f(t, x) = 10[x^2(x - 1)^2y^2(y - 1)^2 - 2t((6x^2 - 6x + 1)(y^2(y - 1)^2) + (6y^2 - 6y + 1)(x^2(x - 1)^2))] \]

\( \mu \in (0, 10] \).

**Figure:** Test with \( L^\infty(0, T; H^1(\Omega)) \) (left) and \( L^\infty(0, T; L^2(\Omega)) \) (right) relative errors with a new parameter \((a, b) = (2, 4)\), \( T = 5 \), \( \Omega = [0, 1] \times [0, 1] \).
\[ f(t, x) = 10[x^2(x - 1)^2y^2(y - 1)^2 - 2t((6x^2 - 6x + 1)(y^2(y - 1)^2) + (6y^2 - 6y + 1)(x^2(x - 1)^2))], \]

\[ \mu \in (0, 10]. \]

<table>
<thead>
<tr>
<th>NIRB rectified error</th>
<th>max</th>
<th>[ | u_h(n\Delta t_F)(\mu) - u_{n^h}^N(n(\mu))|_{\mu_0^1} ]</th>
<th>max</th>
<th>[ | u_h(n\Delta t_F)(\mu) - u_h(n\Delta t_F)(\mu)|_{\mu_0^1} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>2.31 \times 10^{-10}</td>
<td>6.84</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Maximum \( H^1 \) error over the parameters \([\mu = 10]\) (compared to the true NIRB projection and to the FEM coarse projection) with \( N = 20 \)
NIRB approximations at time n=0, 4, 7, 10
# Numerical results

## Table: FEM runtimes

<table>
<thead>
<tr>
<th>FEM high fidelity solver</th>
<th>FEM coarse solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>00:03</td>
<td>00:02</td>
</tr>
</tbody>
</table>

## Table: NIRB runtimes ($N = 18$)

<table>
<thead>
<tr>
<th>NIRB Offline</th>
<th>classical rectified NIRB online</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:45</td>
<td>00:02</td>
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</tbody>
</table>
Brusselator equations

\[ \partial_t u = a + uv^2 - (b + 1)u + \alpha \Delta u \]
\[ \partial_t v = bu - uv^2 + \alpha \Delta v. \]

- \((u(x, t; \mu), v(x, t; \mu))\): Unknowns

- \(\mu=(a,b,\alpha) \in \mathbb{R}^3\): Variable parameter
Brusselator equations

\[ a = 2, \ b = 4 \in (0, 5). \]

**Figure:** Test with \( L^\infty(0, T; H^1(\Omega)) \) (left) and \( L^\infty(0, T; L^2(\Omega)) \) (right) relative errors with a new parameter \( \mu = 1, \ T = 2, \ \Omega = [0, 1] \times [0, 1] \)
Brusselator equations

\[ a = 2, \quad b = 4 \in (0, 5). \]

<table>
<thead>
<tr>
<th>NIRB rectified error</th>
<th>[ \max_{n=1,\ldots,T/\Delta T_F} | u_h(n\Delta T_F)(\mu) - u^{N,n}<em>h(\mu) |</em>{H^0_0} ]</th>
<th>[ \max_{n=1,\ldots,T/\Delta T_F} | u_h(n\Delta T_F)(\mu) |_{H^1_0} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.3 \times 10^{-9}</td>
<td>4.5</td>
</tr>
</tbody>
</table>

**Table:** Maximum \( H^1 \) error over the parameters \([\mu = 10]\) (compared to the true NIRB projection and to the FEM coarse projection) with \( N = 20 \)
Conclusions & Perspectives

- Error estimates of the NIRB 2-grid method with parabolic problems
- Development of two new NIRB tools

Figure: Meniscus tissue

Perspectives

- Two-grid a-posteriori error estimates
Conclusions & Perspectives

- Error estimates of the NIRB 2-grid method with parabolic problems
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Figure: Meniscus tissue

Perspectives

- Two-grid a-posteriori error estimates

Merci pour votre attention!